Fermionization transform for certain higher-dimensional quantum spin models

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Proposed is a generalization of Jordan-Wigner transform that allows to exactly fermionize a large family of quantum spin Hamiltonians in dimensions higher than one. The key steps are to enlarge the Hilbert space of the original model by adding to it a collection of stand-alone free spins and to use a combination of these auxiliary operators and the lattice spins to construct a proper fermion representation of the physical Hamiltonian. The transform is especially useful for lattice spin Hamiltonians, where two-spin interactions of *XY* type are either absent or exist only within one-dimensional chains and where the chains are coupled via two-spin interactions of Ising type, ring-exchange terms, or more general multispin interactions that involve an even number of spin operators from each chain. Using the proposed fermionization method we provide a simple argument suggesting that a spin Hamiltonian closely related to the ring-exchange model proposed by Paramekanti *et al.*, [Phys. Rev. B **66**, 054526 (2002)] indeed realizes a spin-liquid state.

Quantum magnetism is one of the richest areas of research in condensed-matter physics. It is also one of the most complicated and least understood subjects, primarily because most quantum magnets represent strongly correlated systems, which do not necessarily relate to a simple weakly interacting model that could be studied using perturbative techniques. The phases and low-energy excitations of quantum magnets are nonuniversal and much is determined by energetics. Some insight can be obtained using effective-field theories,[1] which represent low-energy physics of various low-temperature magnetic phases but to obtain exact unambiguous results is rarely possible.

One-dimensional quantum magnets stand out as a notable exception to this state of affairs. In many cases, quantum spin chains can be solved exactly using various nonperturbative methods.[2] Perhaps, the simplest such example is a chain of half-integer spins with nearest-neighbor *XY* interactions, i.e., $\hat{\mathcal{H}}_{XY,1d} = -J\sum_n [\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + \text{H.c.}]$. The nature of the ground state of this spin model can be understood with the help of Jordan-Wigner[3] transformation, proposed back in 1928. Since our Rapid Communication intends to generalize it to some higher-dimensional models, let us reiterate the key general ideas, which of course are well known: To solve a quantum spin Hamiltonian implies to either calculate the partition function or to prove that it is equivalent to a partition function associated with a different Hamiltonian, which we understand well, preferably a Hamiltonian expressed in terms of creation/annihilation operators of some canonical fermions or bosons. The main difficulty in accomplishing this task is due to the fact that the spin operators, $\hat{\sigma}_n^{\pm} = (\hat{\sigma}_n^x \pm i\hat{\sigma}_n^y)/2$, are neither fermions nor bosons because they commute on different sites $[\hat{\sigma}_n^a, \hat{\sigma}_m^b]_- \propto \delta_{nm}$ and anticommute on the same site: $[\hat{\sigma}_n^a, \hat{\sigma}_n^b]_+ = 0$ (here and below Latin indices, $a, b = \pm$). Clearly, no local transformation can "correct" the anticommutation relation, but Jordan and Wigner[3] showed that there exists a nonlocal transform, now bearing their names, that accomplishes just that

$$\hat{f}_n^\dagger = \hat{\sigma}_n^+ \prod_{m<n} \hat{\sigma}_m^z \quad \text{and} \quad \hat{f}_n = \hat{\sigma}_n^- \prod_{m<n} \hat{\sigma}_m^z, \qquad (1)$$

where $\hat{f}_n^\dagger$ and $\hat{f}_n$ are creation/annihilation fermion operators, $[\hat{f}_n, \hat{f}_m^\dagger]_+ = \delta_{nm}$. Since $(\hat{\sigma}_n^z)^2 = \hat{1}$, $(\hat{\sigma}_n^a)^2 = 0$, and $\hat{\sigma}_n^z = 2\hat{\sigma}_n^+ \hat{\sigma}_n^- - \hat{1}$,

the Hamiltonian for the *XY*-spin chain with nearest-neighbor interactions becomes that of a one-dimensional Fermi gas in the language of f fermions Eq. (1). The Heisenberg spin chain, or a more general *XYZ*-spin chain with nearest-neighbor interactions, take the form of interacting fermion models, which we recognize as a Luttinger liquid. These mappings are possible because the infinite products or strings in Eq. (1) collapse into unity operators in all terms of these Hamiltonians.

It is clear that transform in Eq. (1) itself is independent of the underlying spin model and it always produces proper fermion operators out of the original spins. The problem, however, arises when we use these fermions to describe a generic spin Hamiltonian, e.g., a higher-dimensional lattice model[4] or a spin chain with longer-range interactions: the string products there generally do not disappear and lead to a Hamiltonian, which features terms $\propto \hat{f}_n^\dagger \exp[i\pi \sum_{m \in C_n} \hat{f}_m^\dagger \hat{f}_m]$, where C_n is a parameterization- and model-dependent set of sites surrounding a site n. These remaining strings lead in general to various complicated fermion-gauge-theories with constraints. In this Rapid Communication, we suggest that by enlarging the Hilbert space of the original spin model to include "external" auxiliary states, one can fermionize a large class of quantum spin models in such a way that both the gauge/string factors and the new auxiliary operators disappear from the resulting fermion Hamiltonian, which therefore takes the familiar form of an interacting fermion model.

First, we present the general idea of this construction. Consider a lattice spin Hamiltonian, $\hat{\mathcal{H}}_s[\{\hat{\sigma}_n^a\}]$, expressed in terms of the Pauli matrices. Add to the model another Hamiltonian, $\hat{\mathcal{H}}_\tau[\{\hat{\tau}_l\}]$, expressed in terms of some other operators, $\hat{\tau}$, that commute with all lattice spins and that belong to a representation of an algebra, which may or may not be $\mathfrak{su}(2)_k$ (in all examples of this Rapid Communication, we consider auxiliary spin-one-half "particles" only). The full Hamiltonian reads

$$\hat{\mathcal{H}} = \mathcal{H}_s[\{\hat{\sigma}_n^a\}] + \mathcal{H}_\tau[\{\hat{\tau}_l\}]. \qquad (2)$$

The main idea here is to use *both* the original spins *and* the operators of the auxiliary algebra to construct fermion opera-

tors $\hat{f}_n = F[\{\hat{\sigma}_n^a\}, \{\hat{\tau}_l\}]$ with the goal of achieving the simplest possible form of the resulting fermion theory. Furthermore since the auxiliary algebra is needed only to correct the anticommutation relations, the actual Hamiltonian, $\mathcal{H}_\tau$, in Eq. (2) can be assumed to take the simplest possible form, which quite generally implies that $\mathcal{H}_\tau = 0$. Hence the partition function of the original spin model $Z_s = \mathrm{Tr}\, \exp(-\beta \hat{\mathcal{H}}_s)$ is simply given by $Z_s = Z/Z_\tau$, where $Z_\tau = d_\tau$ is the dimensionality of the auxiliary Hilbert space, which is an irrelevant constant independent of any parameters of the physical model or temperature. On the other hand, an unspecified at this stage fermionization transform would lead to the Hamiltonian

$$\hat{\mathcal{H}} = \tilde{\mathcal{H}}_s[\{\hat{f}_n^\dagger\}, \{\hat{f}_n\}, \{\hat{\tau}_l\}] + 0. \tag{3}$$

We show that there exist quantum spin models that can be conveniently fermionized with the help of auxiliary operators, associated with free stand-alone spins. To provide the simplest explicit example, consider first a two-leg spin ladder

$$\hat{\mathcal{H}}_{\mathrm{XY,2leg}} = \sum_{n;l=1,2} \{[J_\parallel(l)\hat{\sigma}^+(n,l)\hat{\sigma}^-(n+1,l) + \mathrm{H.c.}] - B\hat{\sigma}^z(n,l)\}$$
$$+ \sum_{n,m;l_1 \neq l_2} [J_z(n,m)\hat{\sigma}^z(n,l_1)\hat{\sigma}^z(m,l_2) + \mathrm{H.c.}]. \tag{4}$$

Equation (4) describes two *XY*-spin chains in a magnetic field, coupled via Ising-type interactions. Let us now introduce an auxiliary free spin-1/2 particle, $\hat{\tau}$, and define

$$\hat{f}_\uparrow^\pm(n) = \hat{\sigma}^\pm(n,1)\prod_{m<n}[\hat{\sigma}^z(m,1)]\hat{\tau}_x,$$
$$\hat{f}_\downarrow^\pm(n) = \hat{\sigma}^\pm(n,2)\prod_{m<n}[\hat{\sigma}^z(m,2)]\hat{\tau}_y, \tag{5}$$

where here and below we identify $\hat{f}^- \equiv \hat{f}$ and $\hat{f}^+ \equiv \hat{f}^\dagger = (\hat{f}^-)^\dagger$. Since $[\hat{\sigma}^z(n,l)]^2 = [\hat{\tau}^a]^2 = 1$, the inverse transform is simply $\hat{\sigma}^\pm(n,1/2) = \hat{f}_{\uparrow/\downarrow}^\pm(n)\Pi_{m<n}[\hat{\sigma}^z(m,1/2)]\hat{\tau}_{x/y}$. The infinite products in Eq. (5) represent the familiar Jordan-Wigner strings in Eq. (1) that give rise to the desired anticommutation relations within the chains while the last factor restores proper fermion algebra for all operators involved, i.e.,

$$[\hat{f}_{l_1}^{a_1}(n_1), \hat{f}_{l_2}^{a_2}(n_2)]_+ = \delta_{n_1 n_2}\delta_{l_1 l_2}\delta_{a_1,-a_2},$$

where $n_{1,2} \in \mathbb{Z}$ labels sites, $l_{1,2} = 1,2 \equiv \uparrow,\downarrow$ labels legs, and $a_{1,2} = \pm$ distinguishes creation and annihilation operators. Furthermore, since $\hat{\sigma}^z(n,l) = 2\hat{\sigma}^+(n,l)\hat{\sigma}^-(n,l) - 1 = 2\hat{f}_l^\dagger(n)\hat{f}_l(n) - 1$, $\forall\, n,l$, the fermion representation of Eq. (4) does not involve the auxiliary τ operators (cf., Refs. 5 and 6).

$$\tilde{\hat{\mathcal{H}}}_{\mathrm{XY,2leg}} = -\sum_{n;l=\uparrow,\downarrow} \{[J_\parallel(l)\hat{f}_l^\dagger(n)\hat{f}_l(n+1) + \mathrm{H.c.}] + B\hat{f}_l^\dagger(n)\hat{f}_l(n)\}$$
$$+ \sum_{n,m} \{J_z(n,m)[2\hat{f}_\uparrow^\dagger(n)\hat{f}_\uparrow(n) - 1]$$
$$\times [2\hat{f}_\downarrow^\dagger(m)\hat{f}_\downarrow(m) - 1] + \mathrm{H.c.}\}, \tag{6}$$

where we have omitted an unimportant constant. We see that transform Eq. (5) maps the spin model onto that of two spe-

cies of fermions with interactions. Let us note here in passing that the inclusion of ferromagnetic couplings would translate in Eq. (6) into an attractive interaction between the fermions, which is expected to induce pairing correlations at $T \to 0$. In particular, the nearest-neighbor intrachain attraction would select an *s*-wave pairing while a combination of nearest-neighbor and zigzag intrachain couplings Eq. (4) are expected to give rise to a *p*-wave Cooper pairing[7] for fermions in Eq. (6). The zero-temperature ground state of the latter model may be related to the simplest possible topological superconductor.[8]

Let us now take the next simplest step and consider a periodically translated two-leg spin ladder Eq. (4) to form a two-dimensional square lattice consisting of coupled *XY*-spin chains. We now define the following fermion operators associated with the lattice sites, $\mathbf{r} = (n,l)$:

$$\hat{f}^\dagger(n,l) = \hat{\sigma}^+(n,l)\prod_{m<n}[\hat{\sigma}^z(m,l)]\prod_{k<l}(\hat{\tau}_k^y)\hat{\tau}_l^x,$$
$$\hat{f}(n,l) = \hat{\sigma}^-(n,l)\prod_{m<n}[\hat{\sigma}^z(m,l)]\prod_{k<l}(\hat{\tau}_k^y)\hat{\tau}_l^x, \tag{7}$$

where, we introduced a spin-1/2 operator for each leg [labeled in Eq. (7) by indices k and l]. The existence of these new operators does not disturb the anticommutation relations within each chain because for any product $\hat{f}^{a_1}(n_1,l)\hat{f}^{a_2}(n_2,l)$ the "new" τ-strings square up to an identity operator while the "old" σ strings enforce the usual constraints. However, the existence of the τ operators is crucial to produce the right anticommutation relations for operators in different legs, e.g., for $l_1 < l_2$, we obtain

$$[\hat{f}^{a_1}(n_1,l_1), \hat{f}^{a_2}(n_2,l_2)]_+ = \hat{\sigma}^{a_1}(n_1,l_1)\prod_{m_1<n_1}[\hat{\sigma}^z(m_1,l_1)]$$
$$\times \hat{\sigma}^{a_2}(n_2,l_2)\prod_{m_2<n_2}[\hat{\sigma}^z(m_2,l_2)]$$
$$\times \prod_{l_1<k<l_2}(\hat{\tau}_k^y)\hat{\tau}_{l_2}^x[\hat{\tau}_{l_1}^x, \hat{\tau}_{l_1}^y]_+ = 0. \tag{8}$$

Therefore, operators in Eq. (7) are indeed fermions. Per the same arguments as in the two-leg-spin-ladder case Eq. (4), we find that its two-dimensional realization can be fermionized via Eq. (7) in such a way that neither Jordan-Wigner strings, nor τ operators appear in the Hamiltonian. We simply can replace $\hat{\sigma}^+(n,l) \to \hat{f}^\dagger(n,l)$, $\hat{\sigma}^-(n,l) \to \hat{f}(n,l)$, $\hat{\sigma}^z(n,l) = [2\hat{f}^\dagger(n,l)\hat{f}(n,l) - 1]$, and change the sign of the *XY*-interaction term to account for $\hat{\sigma}^+(n,l)\hat{\sigma}^z(n,l) = -\hat{\sigma}^+(n,l)$.

We now can construct a general spin model in two dimensions for which the particular transform Eq. (7) directly applies. Let us reiterate that by "applies," we mean that the resulting fermion Hamiltonian does not contain the τ operators or any phase factors arising from Jordan-Wigner strings. Otherwise, the transform Eq. (7) as well as Eq. (1) always apply in the sense that they produce the fermion operators using the spin operators available. Let us write down the following rather general Hamiltonian associated with a square lattice
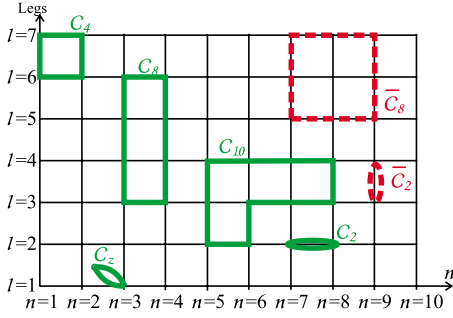
FIG. 1. (Color online) Displayed is a square lattice associated with the general spin model (9) and also shown are allowed and forbidden cycles on the lattice (depicted as loops with solid and dashed outlines correspondingly). Each loop represents a spin interaction term in the Hamiltonian, which involves a product of L $\hat{\sigma}^{\pm}(\mathbf{r})$ operators, where L is the number of sites in the loop, $\mathcal{C}_L$. Shown here are some examples: (i) $\mathcal{C}_2$ corresponds to a nearest-neighbor XY coupling; (ii) $\mathcal{C}_4$ is associated with ring-exchange terms such as in Eq. (11), etc.

$$\hat{\mathcal{H}} = h_z[\{\hat{\sigma}_n^z\}] + \sum_{n,l} [\hat{\sigma}^\alpha(n,l) J_{\alpha\beta}(n,l) \hat{\sigma}^\beta(n+1,l)]$$
$$+ \sum_{\mathcal{C}\subset\text{Even loops}} \sum_{\{a(\mathbf{r})=\pm\}} J[\{a(\mathbf{r})\},\mathcal{C}] \prod_{\mathbf{r}\in\mathcal{C}} \hat{\sigma}^{a(\mathbf{r})}(\mathbf{r}) + \text{H.c.}$$

(9)

and decipher the meaning of each term in Eq. (9). The first term is an arbitrary function of $\hat{\sigma}^z$ operators on each site. The second term describes a collection of decoupled spin chains with arbitrary nearest-neighbor interactions within each chain. Note that a "chain" does not necessarily have to be an actual row or column of a lattice, these legs may be formed by any one-dimensional paths, as long as the collection of these paths covers the entire lattice and no two paths intersect. The last term includes all "allowed" interactions that involve two or more different chains (apart from those interactions, which may already be present in h_z). The indices $a(\mathbf{r})=\pm$ there label either lowering or raising operators, which may in principle appear in any combination, if no global spin conservation is imposed (no number conservation/gauge invariance for the fermions). Finally, $\mathcal{C}$ in Eq. (9) denotes a closed path, or a one cycle, on a lattice: $\mathcal{C}_L=\{\mathbf{r}_1\rightarrow\mathbf{r}_2\rightarrow\mathbf{r}_3\rightarrow\cdots\rightarrow\mathbf{r}_L\}$, where, $\mathbf{r}_i$, represents a lattice site and $|\mathbf{r}_{i+1}-\mathbf{r}_i|=|\mathbf{r}_L-\mathbf{r}_1|=0$ or 1. The label "even loops" in the sum correspond to the cycles that satisfy the following two constraints: (i) each even cycle has an even number of sites present from each leg and (ii) For any site $\mathbf{r}=(n,l)\in\mathcal{C}$ lying in the path, there exists a $\mathbf{r}'=(n',l)\in\mathcal{C}$ from the same leg, which is either the same site ($n'=n$) or its nearest neighbor ($n=n'\pm1$). We note that including the first two terms in Eqs. (9) along with the general third term is in fact redundant because the $\hat{\sigma}^z(\mathbf{r})$ terms can be associated with $\mathcal{C}_z=\{\mathbf{r}\rightarrow\mathbf{r}\}$ cycles (see, Fig. 1) and the nearest-neighbor XY interactions with $\mathcal{C}_2=\{\mathbf{r}\rightarrow\mathbf{r}\pm\mathbf{e}_x\}$. The purpose of all these complicated constraints is simple: we want to avoid both τ operators in the resulting Hamiltonian [recall that $(\hat{\tau}_l^{x,y})^{2n}=1$ hence the choice of even cycles] and any phase

factors that could arise from the Jordan-Wigner string (hence, we eliminate the non-nearest-neighbor interactions within legs[9]). A general Hamiltonian (9) satisfies all these conditions and can be fermionized via the proposed transform in Eq. (7). Figure 1 illustrates the imposed constraints more clearly. The solid-line loops there are examples of allowed paths while the dashed-line loops provide examples of interaction terms, which are "not allowed." Let us note that there may be a simple physical explanation why a Hamiltonian with terms of type $\bar{C}_8$ probably cannot be fermionized via this method, which utilizes auxiliary spins: The loop encircles a site that does not belong to it [$(n=8,\ l=6)$ in Fig. 1] and this site may or may not contain a fermion, which leads to an ambiguity in the phase factors. Hence, the choice of statistics matters in this case while for Hamiltonian (9) it may become largely a question of convenience (cf., Ref. 10).

We can generalize this scheme further by considering more complicated long-range intrachain interactions and various lattice structures including three-dimensional and higher-dimensional lattices (which too can be conveniently fermionized using this method as long as the interaction terms respect the constraints, i.e., belong to the allowed one- and higher-dimensional cycles). However, instead of generalizing Hamiltonian (9), let us on the contrary focus on the following specific realization of it, which involves very interesting ring-exchange terms (associated with loops of type $\mathcal{C}_4$ in Fig. 1)

$$\hat{\mathcal{H}}_{\text{SL}} = \sum_{\mathbf{r}\in\mathbb{Z}^2} \{[J_\parallel\hat{\sigma}^+(\mathbf{r})\hat{\sigma}^-(\mathbf{r}+\mathbf{e}_x) + \text{H.c.}] - B\hat{\sigma}^z(\mathbf{r})\} + \hat{\mathcal{H}}_z + \hat{\mathcal{H}}_4,$$

(10)

where $\hat{\mathcal{H}}_z=J_z\sum_{\mathbf{r}}[\hat{\sigma}^z(\mathbf{r})\hat{\sigma}^z(\mathbf{r}+\mathbf{e}_y)+\text{H.c.}]$, cf. Eq. (4), and

$$\hat{\mathcal{H}}_4 = K_4\sum_{\mathbf{r}} \hat{\sigma}^+(\mathbf{r})\hat{\sigma}^-(\mathbf{r}+\mathbf{e}_x)\hat{\sigma}^+(\mathbf{r}+\mathbf{e}_x+\mathbf{e}_y)\hat{\sigma}^-(\mathbf{r}+\mathbf{e}_y) + \text{H.c.}$$

(11)

is a ring-exchange term. Our method immediately gives the fermionized version of Eq. (10) in the form of an array of Luttinger liquids coupled via density-density and current-current interactions. It is known that such a model hosts a variety of unusual phases, including so-called sliding Luttinger liquids.[11–13] This immediately implies that the underlying *two-dimensional quantum spin model too remains a spin liquid* (that can be dubbed a "Luttinger spin liquid") in the corresponding parameter range. Note that Eq. (10) is closely related to the model proposed earlier by Paramekanti *et al.*,[14] as a realization of a spin/Bose liquid. Indeed, the rotor Hamiltonian of Ref. 14, $\mathcal{H}=U\sum_{\mathbf{r}}(\hat{n}_{\mathbf{r}}-\bar{n})^2 - K\sum_{\mathbf{r}}\cos(\Delta_{xy}\hat{\phi}_{\mathbf{r}})$, reduces in the hardcore-boson limit to a spin-(1/2) model of type Eq. (10) with interchain hopping terms absent (i.e., $J_\parallel=0$). Since the latter model can be fermionized as shown, the appearance of a liquid phase predicted in Ref. 14 is quite natural. Let us also note that model (10) is related to a Bose-metal phase introduced by Motrunich and Fisher.[15] The proposed spin/Bose-metal Hamiltonian is of the following type $\hat{\mathcal{H}}_{\text{BM}}=J\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\hat{\boldsymbol{\sigma}}(\mathbf{r})\cdot\hat{\boldsymbol{\sigma}}(\mathbf{r}')+\hat{\mathcal{H}}_4$, where $\langle\mathbf{r},\mathbf{r}'\rangle$ corresponds to near-

est neighbors on a square lattice or a spin ladder.[15,16] However, the presence of two-dimensional *XY* couplings in the generic Bose-metal Hamiltonian does not allow us to use the fermionization transform in Eq. (7).

Let us note that Jordan-Wigner-type fermionization with or without external operators, when applied to two- and higher-dimensional isotropic *XY* models would produce a gauge theory for the fermions as opposed to a simple interacting theory, such as in Eq. (5). However, there may exist an alternative way to fermionize the *XY* model in terms of BCS fermions with spin instead of spinless Jordan-Wigner fermions. It is known that the partition function of the Richardson model[17,18] of interacting fermions with the Hamiltonian, $\hat{\mathcal{H}}_R = \sum_{\mathbf{r},s=\uparrow,\downarrow} B_{\mathbf{r}} \hat{c}^\dagger_{\mathbf{r},s} \hat{c}_{\mathbf{r},s} + \sum_{\mathbf{r},\mathbf{r}'} J(\mathbf{r},\mathbf{r}') \hat{c}^\dagger_{\mathbf{r},\uparrow} \hat{c}^\dagger_{\mathbf{r},\downarrow} \hat{c}_{\mathbf{r}',\downarrow} \hat{c}_{\mathbf{r}',\uparrow}$ can be mapped onto a combination partition functions, $Z_R = \sum(Z_{spin} \times Z_{p-spin})$, of real spins on singly occupied sites and Anderson[19] pseudospins on paired/empty sites, with Z_{p-spin} associated with an *XY* model for Anderson pseudospins, $\hat{\mathcal{H}}_{XY} = \sum_{\mathbf{r},\mathbf{r}'} J(\mathbf{r},\mathbf{r}') [\hat{\sigma}^+_{\mathbf{r}} \hat{\sigma}^-_{\mathbf{r}'} + \text{H.c.}]$. Hence, by adding to Richardson Hamiltonian nonlinear terms that eliminate single occupancy, we can suppress the real-spin-sector and thereby fermionize the remaining *XY* model associated with the pseudospins.

The general proposed scheme of using external operators to build a fermionized version of a spin model does not necessarily need to involve the particular transform Eq. (7), which should be viewed as merely an example. Furthermore, there always exist multiple different parameterizations or numberings of both the external operators and "internal" Jordan-Wigner-type strings that give rise to the same Hamiltonian and associated partition function (modulo an overall constant). To provide a final example in this context, let us consider a three-leg spin ladder, forming a triangular lattice, see Fig. 2. To fermionize this model, we in addition to transform Eq. (7), can use, e.g., the following fermion operators, $\hat{f}^\pm_l(n) = \hat{\sigma}^\pm(n,1) \Pi_{m<n} [\hat{\sigma}^z(m,1)] \hat{\tau}^l$, where $l=1,2,3 \equiv x,y,z$ is
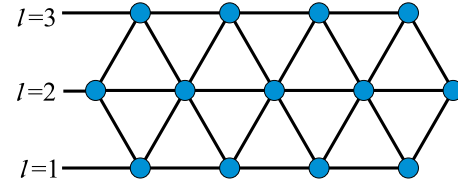

FIG. 2. (Color online) An example of a three-leg spin ladder, which can be fermionized using a transform that involves just one external spin operator (see text for details) in contrast to Eq. (7).

the leg index. Hence, we can fermionize this three-leg ladder using just one auxiliary spin, $\hat{\tau}$. However, whether or not this particular parameterization is useful depends on a Hamiltonian of interest. If only nearest-leg ring exchanges are included, this transform leads to a proper fermion Hamiltonian, which does not involve the external spin. However, the inclusion of center-of-mass conserving ring exchanges that involve all three legs (such as present in the models discussed in Refs. 20 and 21) would lead to the appearance of $\hat{\tau}^y$-"gauge" factors in the fermion Hamiltonian.

Finally, we note that the fact that the proposed transform works only for a very restricted class of models reflects the fact that the number of quantum spin models, whose partition function is proportional to that of a fermion model is limited. However, there exist larger classes of spin models whose partition function is expressible as a linear combination or a series of fermionic partition functions. A very promising approach to analyze these models is to allow auxiliary particles that are not $\mathfrak{su}(2)$ spins but are described by representations of other (arbitrary auxiliary algebras), which may be used in conjunction with physical spins to build fermionic representations. Examples of the use of this generalized approach will be presented elsewhere.

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